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ZZ brane amplitudes from matrix models

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Abstract: We study instanton contribution to the partition function of the one matrix model in the k-th multicritical region, which corresponds to the $(2, 2k - 1)$ minimal model coupled to Liouville theory. The instantons in the one matrix model are given by local extrema of the effective potential for a matrix eigenvalue and identified with the ZZ branes in Liouville theory. We show that the 2-instanton contribution in the partition function is universal as well as the 1-instanton contribution and that the connected part of the 2-instanton contribution reproduces the annulus amplitudes between the ZZ branes in Liouville theory. Our result serves as another nontrivial check on the correspondence between the instantons in the one matrix model and the ZZ branes in Liouville theory, and also suggests that the expansion of the partition function in terms of the instanton numbers are universal and gives systematically ZZ brane amplitudes in Liouville theory.

Keywords: D-branes, 2D Gravity, Matrix Models.

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Contents

1. Introduction

The nonperturbative study of noncritical strings in terms of the matrix models [1, 2, 3] led to an important suggestion [4] that string theory in general possesses the nonperturbative effect that behaves as $\sim e^{-\frac{1}{g_s}}$. It was pointed out in [5] and is now widely believed that this effect is attributed to D-branes. Indeed, the discovery of D-branes in Liouville theory which are called the FZZT branes [6, 7, 8] and the ZZ branes [9] triggered recent progress in noncritical strings and the matrix models, in which the origin of the nonperturbative effect $e^{-\frac{1}{g_s}}$ given by the string equations of the matrix models [4, 10, 11] was identified with the ZZ branes [12]–[16].

The authors of ref. [17] studied intensively the D-branes in a series of noncritical strings, the (p,q) minimal conformal field theory coupled to two-dimensional quantum gravity (Liouville theory).[1] They call this the (p,q) minimal string theory. They showed that there exist $\frac{(p-1)(q-1)}{2}$ independent ZZ branes labeled by (m,n), where $qm-pn>0$, and those ZZ branes correspond to the singularities of an auxiliary Riemann surface that are formed by the analytically continued boundary cosmological constant and the derivative of the FZZT disk amplitude with respect to the boundary cosmological constant. In particular, when $(p,q)=(2,2k-1)$, the minimal string theory is realized by the k-th multicritical region of the one matrix model [18] and the ZZ branes are identified with the local extrema of

[1] Here, p and q are relatively prime integers and $p<q$.

the effective potential for a matrix eigenvalue [17, 19]. These extrema can be called the instantons. The annulus amplitudes between the D-branes in the minimal string theory were evaluated in [20] to examine the deformations by the D-branes.

The authors of ref. [19] explored in detail the nonperturbative effect stemming from the ZZ brane in $c = 0$ noncritical string theory (the $(2,3)$ minimal string theory) from the viewpoints of the one matrix model as well as of the loop equations (the string field theory). One of the important results of ref. [19] is that the ratio of the 1-instanton sector in the partition function of the matrix model to the 0-instanton sector is universal, namely, it does not depend on the detailed structure in the potential of the matrix model. Actually, they confirmed this up to next to leading order in the $1/N$ expansion by the explicit calculation. This ratio is interpreted as the chemical potential of the instanton in the dilute gas approximation, where the interaction between the instantons is turned off. The results of [10] and [16] tell us that the leading order of this ratio given by e^{-c/g_s} is equal to $e^{Z_{ZZ}}$, where Z_{ZZ} is the ZZ disk amplitude and $Z_{ZZ} \sim 1/g_s$. The extension of this analysis to the supersymmetric case ($\hat{c} = 0$ type 0B string theory) was reported very recently [21].

In this paper, we turn on the interaction between the instantons in the one matrix model. In general, the interaction between D-branes is given by the annulus amplitude between the D-branes. If the identification of the instantons in the matrix model with the ZZ branes in Liouville theory is valid, we should be able to obtain the annulus amplitudes between the ZZ branes by considering the interaction between the instantons. For this purpose, we need to consider the general $(2, 2k-1)$ case. The reason is as follows. The interaction between the identical instantons diverges due to the Vandermonde determinant for the matrix eigenvalues. Correspondingly, the annulus amplitudes between the identical ZZ branes also diverges. On the other hand, the annulus amplitudes between the different ZZ branes is finite. The $(2, 2k-1)$ theory possesses $k-1$ independent instantons in the one matrix model and the $k-1$ independent ZZ branes in Liouville theory. So, the $(2,3)$ case is not sufficient for our purpose.

First, we generalize the calculation of the ratio of the 1-instanton sector to the 0-instanton sector in [19] to the $(2, 2k-1)$ case. We show that it is also universal for generic k and the leading order is given by the ZZ brane disk amplitude. Next, we consider the 2-instanton sector in the partition function of the matrix model turning on the interaction between the instantons. We show that the ratio of the 2-instanton sector to the 0-instanton sector is universal and its connected part indeed reproduces the annulus amplitudes between the ZZ branes in Liouville theory. Our result serves as another nontrivial check on the correspondence between the instantons in the one matrix model and the ZZ branes in Liouville theory, and also suggests that the expansion of the partition function in terms of the instanton numbers are universal and gives systematically the ZZ brane amplitudes in Liouville theory.

This paper is organized as follows. Sections 2 and 3 are devoted to the reviews of the ZZ branes in the $(2, 2k-1)$ minimal string theory and of the k-th multicritical region of the one matrix model, respectively. In section 4, we see the behavior of the effective potential of a matrix eigenvalue and define the expansion of the partition function in the instanton numbers. Sections 5 and 6 are the main part of this paper. We perform the above mentioned calculations of the ratio of the 1-instanton sector to the 0-instanton sector in section 5 and of the ratio of the 2-instanton sector to the 0-instanton sector in section 6. Section 7 is devoted to summary and discussion. In appendix A, we give the detailed calculation for the $(2,5)$ case. In appendix B, we gather some formulae used in the main text.

2. ZZ branes in Liouville theory

In this section, we describe a part of the results in refs. [17, 20] which are relevant for our purpose. Many of the features of the (p,q) minimal string theory are provided by an auxiliary Riemann surface $\mathcal{M}_{p,q}$, which is described by the algebraic equation

$$F(\xi,\eta) = T_q(\xi) - T_p(\eta) = 0\,, \tag{2.1}$$

where $T_p(\cos\theta) = \cos p\theta$ is the Chebyshev polynomial of the first kind. ξ is the ratio of the boundary cosmological constant ζ to the square root of the bulk cosmological constant μ while η is proportional to the derivative of the FZZT disk amplitude Z_{FZZT} with respect to ζ.

$$\xi = \frac{\zeta}{\sqrt{\mu}}\,, \qquad \eta \sim \mu^{-\frac{q}{2p}}\partial_\zeta Z_{FZZT}\,. \tag{2.2}$$

It is convenient to introduce auxiliary parameters σ and $z = \cosh\frac{\pi\sigma}{\sqrt{pq}}$, in terms of which

$$\begin{aligned} \xi &= \cosh\pi\sqrt{\frac{p}{q}}\sigma = T_p(z)\,, \\ \eta &= \cosh\pi\sqrt{\frac{q}{p}}\sigma = T_q(z)\,. \end{aligned} \tag{2.3}$$

Note that z covers the surface exactly once. The ZZ branes correspond to the singularities of $\mathcal{M}_{p,q}$ given by $F = \partial_\xi F = \partial_\eta F = 0$, which correspond to two different values of z denoted by $z^\pm$.

In what follows, we restrict ourselves to the case in which $(p,q) = (2,2k-1)$. In this case (2.1) is given by

$$2\eta^2 = T_{2k-1}(\xi) + 1 = \frac{(T_k(\xi) + T_{k-1}(\xi))^2}{\xi+1}\,. \tag{2.4}$$

We define $\eta_k(\xi)$ by

$$\sqrt{2}\eta_k(\xi) = \frac{T_k(\xi) + T_{k-1}(\xi)}{\sqrt{\xi+1}}\,. \tag{2.5}$$

Then, $F = \partial_\xi F = \partial_\eta F = 0$ are equivalent to

$$\eta_k = 0\,, \qquad \partial_\xi \eta_k(\xi)^2 = 0\,. \tag{2.6}$$

These equations are solved as

$$\xi_n = -\cos\frac{2\pi n}{2k-1}\,, \qquad n = 1,2,\ldots,k-1\,, \tag{2.7}$$

which corresponds to

$$z_n^{\pm} = -\sin\frac{\pm\pi n}{2k-1}\,. \tag{2.8}$$

ξ_n characterizes the $(1,n)$ ZZ brane. η_k is expressed in terms of ξ_n's as

$$2\eta_k(\xi)^2 = 2^{2k-2}(\xi-\xi_1)^2(\xi-\xi_2)^2\cdots(\xi-\xi_{k-1})^2(\xi+1)\,. \tag{2.9}$$

It is convenient to introduce an integral of η_k.

$$v_k(\xi) \equiv \int^{\xi} d\xi'\sqrt{2}\eta_k(\xi') = \frac{T_{k+1}(\xi)+T_k(\xi)}{(2k+1)\sqrt{\xi+1}} - \frac{T_{k-1}(\xi)+T_{k-2}(\xi)}{(2k-3)\sqrt{\xi+1}}\,. \tag{2.10}$$

This is proportional to the FZZT disk amplitude: $v_k(\xi) \sim \mu^{-\frac{k}{2}-\frac{1}{4}}Z_{FZZT}$. Later, we will use the following quantities which are proportional to the ZZ brane disk amplitudes.

$$v_k(\xi_n) = (-1)^{k+n}\sqrt{2}\left(\frac{1}{2k+1}+\frac{1}{2k-3}\right)\sin\frac{2\pi n}{2k-1}\,. \tag{2.11}$$

The annulus amplitudes between the ZZ branes was calculated in [20]. The result for that between the $(1,n)$ and $(1,n')$ ZZ branes is

$$Z_{n,n'} = \log\frac{(z_n^+ - z_{n'}^+)(z_n^- - z_{n'}^-)}{(z_n^+ - z_{n'}^-)(z_n^- - z_{n'}^+)}\,. \tag{2.12}$$

3. The k-th multicritical region of the one matrix model

We are concerned with the one matrix model with a generic potential.

$$\begin{aligned} Z &= \int d\phi\, e^{-N\,\mathrm{tr}\,V(\phi)}\,, \\ V(x) &= \frac{1}{2}x^2 - \sum_{m=3}^{\infty}\frac{g_m}{m}x^m\,, \end{aligned} \tag{3.1}$$

where ϕ is an $N\times N$ hermitean matrix. By diagonalizing ϕ, this integral is reduced to

$$Z = \int\prod_{i=1}^{N} d\lambda_i\,\Delta_N(\lambda_1,\ldots,\lambda_N)^2 e^{-N\sum_{i=1}^{N}V(\lambda_i)}\,, \tag{3.2}$$

where $\lambda_1,\ldots,\lambda_N$ are eigenvalues of ϕ and $\Delta_N(\lambda_1,\ldots,\lambda_N)$ is the Vandermonde determinant in terms of $\lambda_1,\ldots,\lambda_N$. It is a standard technique to introduce the orthogonal polynomials $P_n(x)$, which satisfy

$$\int dx\, e^{-NV(x)}P_m(x)P_n(x) = h_n\delta_{mn}\,, \tag{3.3}$$

where $P_n(x)$ is a polynomial of degree n and is normalized so that the coefficient of x^n equals one. It is easy to see that the following recursion relation holds.

$$xP_n(x) = P_{n+1}(x) + s_nP_n(x) + r_nP_{n-1}(x)\,, \tag{3.4}$$

$$r_n = \frac{h_n}{h_{n-1}}\,. \tag{3.5}$$

The partition function Z_N is expressed in terms of r_n:

$$Z = N! h_0 h_1 \cdots h_{N-1} = N! h_0^N \prod_{n=1}^{N-1} r_n^{N-n} . \tag{3.6}$$

Then, the relevant part of the free energy $F = \log Z$ takes the form

$$F = \sum_{n=1}^{N-1} (N-n) \log r_n . \tag{3.7}$$

The Schwinger-Dyson equations for the orthogonal polynomials,

$$\begin{aligned} n h_{n-1} &= \int dx\, e^{-NV(x)} \frac{dP_n(x)}{dx} P_{n-1}(x) \\ 0 &= \int dx\, \frac{d}{dx} (e^{-NV(x)} P_n(x) P_n(x)) , \end{aligned} \tag{3.8}$$

give recursion relations for r_n and s_n, from which one can determine r_n and s_n as functions of g_m's.

We need the k-th multicritical region of the one matrix model to obtain the $(2, 2k-1)$ minimal string theory. The k-th multicritical region is realized by fine-tuning $k-1$ parameters among g_m's and taking $N \to \infty$ limit. We introduce a continuous variable $\sigma = n/N$ in order to examine the critical behavior of the model. The critical point corresponds to $\sigma = 1$. As is explained in appendix C of ref. [19], if the functions $r(\sigma)$ and $s(\sigma)$ are defined by

$$\begin{aligned} r_n &= r(\sigma) , \\ s_n &= s\left(\sigma + \frac{1}{2N}\right) , \end{aligned} \tag{3.9}$$

$r(\sigma)$ and $s(\sigma)$ are $\mathcal{O}(N^0)$ quantities and the corrections start with $\mathcal{O}(\frac{1}{N^2})$. Moreover, the $\mathcal{O}(N^0)$ parts of $r(\sigma)$ and $s(\sigma)$ behave at the critical point like

$$\frac{\partial r/\partial\sigma}{\partial s/\partial\sigma} = \sqrt{r_c} , \tag{3.10}$$

where r_c is the critical value of r_n. Now we are ready to write down the scaling limit of the one matrix model which gives rise to a perturbation around the k-th critical point and corresponds to the $(2, 2k-1)$ minimal string theory [18]:

$$\begin{aligned} g_{m_i} &= g_{m_i c}(1 - \beta_{m_i} \mu \varepsilon^2) , \qquad i = 1, \ldots, k-1 , \\ r(\sigma) &= r_c \left(1 - \frac{1}{2}\alpha\varepsilon u(\tau)\right) , \\ s(\sigma) &= s_c - \frac{1}{2}\alpha\sqrt{r_c}\varepsilon u(\tau) , \\ \sigma &= 1 - \varepsilon^k \nu\tau , \\ \frac{1}{N} &= \varepsilon^{k+\frac{1}{2}} \kappa g_s , \end{aligned} \tag{3.11}$$

where ε is a cutoff so that $\varepsilon \to 0$ corresponds to the continuum limit. $g_{m_i c}$, r_c and s_c are critical values of g_{m_i}, r and s, respectively, which are dependent on the detailed structure in the potential of the matrix model. μ is the bulk cosmological constant which is identified with that in the Liouville theory. α, β_{m_i}, ν and κ are certain constants. In (3.11), we restrict ourselves to the leading order of the $1/N$ expansion and have taken (3.10) into account. α, β_{m_i} and ν in (3.11) are adjusted in such a way that $u(\sigma)$ obeys a string equation [18]

$$\sum_{j=0}^{\infty} t_j u^j = \tau \,, \tag{3.12}$$

where

$$\begin{aligned} t_{k-2p} &= C_{k-2p}\mu^p \,, \qquad p = 0, 1, \ldots, \left[\frac{k}{2}\right], \\ C_{k-2p} &= \frac{(-1)^{k+1}\pi}{\sqrt{8}} \frac{2^{k-2p}}{(k-2p)!p!\Gamma(p-k+\frac{3}{2})}\,, \qquad \text{other } t_j = 0\,. \end{aligned} \tag{3.13}$$

This represents the above mentioned perturbation around the k-th multicritical point. When $\tau = 0$, the string equation (3.12) allows a solution

$$u(0) = \sqrt{\mu}\,. \tag{3.14}$$

The universal part of the sphere contribution to the free energy (3.7) is expressed by $u(\tau)$ as

$$F^{(\text{sphere})} = N^2 \int_0^1 d\sigma\,(1-\sigma)\log r(\sigma) = \frac{1}{2}\kappa^{-2}\nu^2\alpha g_s^{-2} \int^0 d\tau\, \tau u(\tau)\,. \tag{3.15}$$

In the following sections, we use the resolvent, which is defined in the large-N limit (the leading order of the $1/N$ expansion) by

$$R(x) = \left\langle \frac{1}{N}\,\mathrm{tr}\left(\frac{1}{x-\phi}\right)\right\rangle \tag{3.16}$$

$R(x)$ is related to the eigenvalue density $\rho(x)$ as

$$\rho(x) = -\frac{1}{\pi}\mathrm{Im}R(x+i0)\,. \tag{3.17}$$

By solving the loop equation, the form of $R(x)$ is determined as

$$\begin{aligned} R(x) &= \frac{1}{2}V'(x) + W(x)\,, \\ W(x) &= \frac{1}{2}\sqrt{V'(x)^2 + p(x)}\,, \end{aligned} \tag{3.18}$$

where $p(x)$ is a polynomial of degree $m_0 - 2$ when

$$g_{m_0} \neq 0\,, \qquad g_m = 0 \qquad \text{for } m > m_0\,. \tag{3.19}$$

$p(x)$ is determined by the structure of the cut, namely the location where the eigenvalues are distributed, and the condition that $R(x) \sim \frac{1}{x}$ when $|x| \to \infty$. We are interested in the one-cut solution, in which $\rho(x)$ is nonzero only for the period $[b, a]$. Then $W(x)$ takes the form

$$W(x) = \frac{1}{2}K(x)(x - x_1)(x - x_2)\cdots(x - x_{k-1})\sqrt{(x-a)(x-b)}\,, \tag{3.20}$$

where $K(x)$ is a polynomial of degree $m_0 - k - 1$. In the scaling limit (3.11), a and x_n behave like

$$\begin{aligned} a &= x_*(1 - \chi_a\sqrt{\mu}\varepsilon)\,, \\ x_n &= x_*(1 + \chi_n\sqrt{\mu}\varepsilon)\,, \qquad n = 1, \ldots, k-1\,, \end{aligned} \tag{3.21}$$

where χ_a and χ_n are given constants. On the other hand, one can regard b and $K(x)$ as some constants in the scaling limit. If x is scaled in the scaling limit (3.11) as

$$x = x_*(1 + \varepsilon\tilde{\zeta})\,, \tag{3.22}$$

$W(x)$ starts with a term proportional to $\varepsilon^{k-\frac{1}{2}}$ which is the universal part of the resolvent. We will see in section 5 that $\chi_n = \xi_n\chi_a$ and $\chi_a = \alpha x_*^{-1} r_c^{\frac{1}{2}}$ for generic $V(x)$, $\chi_a = \frac{1}{4}\alpha$ for even $V(x)$. Hence, if $\tilde{\zeta}$ is tuned as $\tilde{\zeta} = \chi_a\zeta$, the universal part of the resolvent becomes proportional to the derivative of the FZZT disk amplitude with respect to the boundary cosmological constant, namely $\eta(\xi)$. This is anticipated because the resolvent is interpreted as the expectation value of a marked macroscopic loop in the matrix model and the macroscopic loop is nothing but the geometrical meaning of the FZZT brane. In appendix A, we illustrate the calculations in this section with the case in which $k = 3$ and the potential is even. We see that $\chi_n = \xi_n\chi_a$ and $\chi_a = \frac{1}{4}\alpha$ actually hold.

4. The effective potential for an eigenvalue and instantons

We consider the situation in which a single eigenvalue, say λ_N, is separated from the others. The partition function (3.2) is expressed as

$$Z = \int dx \int \prod_{i=1}^{N-1} d\lambda_i \left(\prod_{i=1}^{N-1}(x - \lambda_i)^2\right) \Delta_{N-1}(\lambda_1, \ldots, \lambda_{N-1})^2\, e^{-N\sum_{i=1}^{N-1}V(\lambda_i)} e^{-NV(x)}\,, \tag{4.1}$$

where we set $\lambda_N = x$. By using an $(N-1)\times(N-1)$ hermitean matrix ϕ_{N-1}, this is rewritten as

$$Z = Z_{N-1}\int dx \langle \det(x - \phi_{N-1})^2\rangle_{N-1} e^{-NV(x)}\,, \tag{4.2}$$

where

$$\begin{aligned} Z_{N-1} &= \int d\phi_{N-1}\, e^{-N\,\mathrm{tr}\,V(\phi_{N-1})}\,, \\ \langle\mathcal{O}\rangle_{N-1} &= \frac{1}{Z_{N-1}}\int d\phi_{N-1}\,\mathcal{O}\, e^{-N\,\mathrm{tr}\,V(\phi_{N-1})}\,. \end{aligned} \tag{4.3}$$

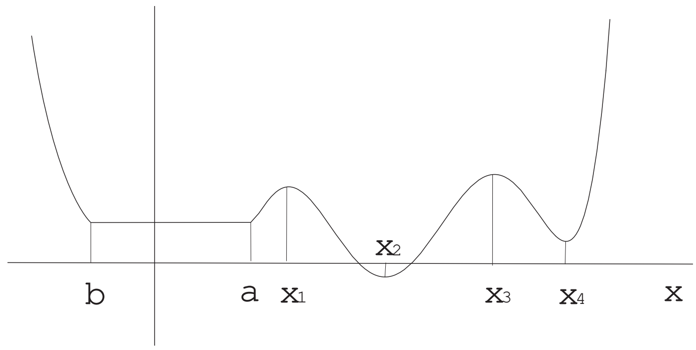

Figure 1: Effective potential for an eigenvalue in the $k = 5$ case.

The effective potential for x is defined by

$$V_{\text{eff}}(x) = V(x) - \frac{1}{N} \log\langle \det(x - \phi_{N-1})^2 \rangle_{N-1} \tag{4.4}$$

in such a way that

$$Z = Z_{N-1} \int dx \, e^{-NV_{\text{eff}}(x)}. \tag{4.5}$$

At leading order of the $1/N$ expansion, the following calculation is justified.

$$\begin{aligned} \langle \det(x - \phi_{N-1})^2 \rangle_{N-1} &= \langle \det(x - \phi)^2 \rangle \\ &= \exp\left[2\mathrm{Re}\langle \mathrm{tr} \log(x - \phi) \rangle\right] \\ &= \exp\left[2N\mathrm{Re} \int^x dx' \, R(x')\right]. \end{aligned} \tag{4.6}$$

Therefore, using (3.18), we find that the leading order of $V_{\text{eff}}(x)$ is

$$V_{\text{eff}}^{(0)}(x) = -2\mathrm{Re} \int^x dx' \, W(x'). \tag{4.7}$$

Or equivalently, $V_{\text{eff}}^{(0)\prime}(x) = -2\mathrm{Re}W(x)$. Then, we see from (3.20) that $V_{\text{eff}}^{(0)}(x)$ is constant in the cut and that $V_{\text{eff}}^{(0)}(x)$ takes local extrema at $x = x_1, \ldots, x_{k-1}$. We ignore extrema coming from $K(x)$ in (3.20), since they do not contribute in the scaling limit. In figure 1, we draw the shape of $V_{\text{eff}}^{(0)}(x)$ roughly in the $k = 5$ case.

These extrema can be considered as the instantons in the one matrix model and will be identified with the ZZ branes in Liouville theory. We label by $\{q_1, \ldots, q_{k-1}\}$ the configuration in which q_n $(n = 1, \ldots, k-1)$ eigenvalues among N are located around $x = x_n$ and the other $N - q$ eigenvalues are located in the cut, where $q = q_1 + \cdots + q_{k-1}$. Namely, q_n is the instanton number of the n-th instanton. We denote by $\int_{x_n} dx$ the perturbative expansion around the 'classical solution' $x = x_n$, which yields the $1/N$ expansion. The leading and subleading contributions to this expansion are nothing but the saddle point integral over x around $x = x_n$. We expand the partition function in terms of the instanton numbers as follows.

$$\begin{aligned} Z &= \sum_{q_1,\ldots,q_{k-1}=0}^{\infty} Z^{\{q_1,\ldots,q_{k-1}\}} \\ &= Z^{(\text{0-inst.})} \sum_{q_1,\ldots,q_{k-1}=0}^{\infty} \mathcal{A}^{\{q_1,\ldots,q_{k-1}\}} , \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} Z^{\{q_1,\cdots,q_{k-1}\}} &= \frac{N!}{q_1!\cdots q_{k-1}!(N-q)!} \left(\prod_{n=1}^{k-1} \int_{x_n} \prod_{i_n=1}^{q_n} dx_{i_n}^{(n)} \right) \int_{b\le\lambda_i\le a} \prod_{i}^{N-q} d\lambda_{i=1} \times \\ &\quad \times \Delta_q(x_1^{(1)},\cdots,x_{q_1}^{(1)},x_1^{(2)},\cdots,x_{q_2}^{(2)},\cdots,x_1^{(k-1)},\cdots,x_{q_{k-1}}^{(k-1)})^2 \times \\ &\quad \times \left(\prod_{n=1}^{k-1} \prod_{i_n=1}^{q_n} \prod_{i=1}^{N-q} (x_{i_n}^{(n)} - \lambda_i)^2 \right) \Delta_{N-q}(\lambda_1,\cdots,\lambda_{N-q})^2 \times \\ &\quad \times e^{-N\sum_{i=1}^{N-q} V(\lambda_i)}\, e^{-N\sum_{n=1}^{k-1}\sum_{i_n=1}^{q_n} V(x_{i_n}^{(n)})} , \\ Z^{(\text{0-inst.})} &= Z^{\{0,\cdots,0\}} = \int_{b\le\lambda_i\le a} \prod_{i=1}^{N} d\lambda_i\, \Delta_N(\lambda_1,\cdots,\lambda_N)^2\, e^{-N\sum_{i=1}^{N} V(\lambda_i)} , \\ \mathcal{A}^{\{q_1,\cdots,q_{k-1}\}} &= \frac{Z^{\{q_1,\cdots,q_{k-1}\}}}{Z^{(\text{0-inst.})}} , \\ \mathcal{A}^{\{0,\cdots,0\}} &= 1 . \end{aligned} \tag{4.9}$$

For example,

$$\begin{aligned} Z^{\{0,\cdots,0,q_n=1,0,\ldots,0,q_{n'}=1,0,\ldots,0\}} &= N(N-1) \int_{x_n} dx \int_{x_{n'}} dy \int_{b\le\lambda_i\le a} \prod_{i=1}^{N-2} d\lambda_i \times \\ &\quad \times (x-y)^2 \left(\prod_{i=1}^{N-2} (x-\lambda_i)^2 (y-\lambda_i)^2 \right) \Delta_{N-2}(\lambda_1,\ldots,\lambda_{N-2})^2 \times \\ &\quad \times e^{-N\sum_{i=1}^{N-2} V(\lambda_i)}\, e^{-NV(x)-NV(y)} . \end{aligned} \tag{4.10}$$

We make the following abbreviations for the quantities which we are concerned with in subsequent sections.

$$\begin{aligned} \mathcal{A}^{\{0,\cdots,0,q_n=1,0,\cdots,0\}} &= \frac{Z^{\{0,\cdots,0,q_n=1,0,\cdots,0\}}}{Z^{(\text{0-inst.})}} = \mathcal{A}^{(n)} , \\ \mathcal{A}^{\{0,\cdots,0,q_n=2,0,\cdots,0\}} &= \frac{Z^{\{0,\cdots,0,q_n=2,0,\cdots,0\}}}{Z^{(\text{0-inst.})}} = \mathcal{A}^{(n,n)} , \\ \mathcal{A}^{\{0,\cdots,0,q_n=1,0,\cdots,0,q_{n'}=1,0,\cdots,0\}} &= \frac{Z^{\{0,\cdots,0,q_n=1,0,\cdots,0,q_{n'}=1,0,\cdots,0\}}}{Z^{(\text{0-inst.})}} = \mathcal{A}^{(n,n')} . \end{aligned} \tag{4.11}$$

$\mathcal{A}^{(n)}$ is the ratio of the 1-instanton sector to the 0-instanton sector while $\mathcal{A}^{(n,n)}$ and $\mathcal{A}^{(n,n')}$ are the ratios of the 2-instanton sectors to the 0-instanton sector. We make $\mathcal{A}^{(n,n')}$ represent

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both the cases, $n = n'$ and $n \neq n'$. Our calculations in the following sections are based on the $1/N$ expansion. Note that the expansion is not in term of $1/N^2$ but in terms of $1/N$ due to the instanton effects. It will turn out that $\log \mathcal{A}^{(n)}$ and $\log \mathcal{A}^{(n,n')}$ start with $\mathcal{O}(N)$. We will evaluate $\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms, which become $\mathcal{O}(1/g_s)$, $\mathcal{O}(\log g_s)$ and $\mathcal{O}(g_s^0)$ terms in the continuum limit, respectively.

5. 1-instanton sectors and the ZZ brane disk amplitudes

In this section, we calculate $\mathcal{A}^{(n)}$. $\mathcal{A}^{(n)}$ will turn out to start with $\mathcal{O}(N)$. We will evaluate $\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms in $\mathcal{A}^{(n)}$. We will show that these terms are universal and the leading order term (the $\mathcal{O}(N)$ term) agrees with the $(1,n)$ ZZ brane disk amplitude. We first write down the definition of $\mathcal{A}^{(n)}$.

$$\begin{aligned}
\mathcal{A}^{(n)} &= \frac{Z^{\{0,\cdots,0,q_n=1,0,\cdots,0\}}}{Z^{(\text{0-inst.})}} \\
&= \frac{1}{Z^{(\text{0-inst.})}} N \int_{x_n} dx \int_{b\leq<\lambda_i\leq a} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} (x-\lambda_i)^2 \Delta_{N-1}(\lambda_1,\cdots,\lambda_{N-1})^2 \times \\
&\quad \times e^{-N\sum_{i=1}^{N-1} V(\lambda_i)}\, e^{-NV(x)} \\
&= N \frac{Z_{N-1}^{(\text{0-inst.})}}{Z^{(\text{0-inst.})}} \int_{x_n} dx\, \langle \det(x-\phi_{N-1})^2 \rangle_{N-1}^{(\text{0-inst.})}\, e^{-NV(x)}\,,
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
Z_{N-1}^{(\text{0-inst.})} &= \int_{b\leq\lambda_i\leq a} \prod_{i=1}^{N-1} d\lambda_i\, \Delta_{N-1}(\lambda_1,\cdots,\lambda_{N-1})^2\, e^{-N\sum_{i=1}^{N-1} V(\lambda_i)}\,, \\
\langle \det(x-\phi_{N-1})^2 \rangle_{N-1}^{(\text{0-inst.})} &= \frac{1}{Z_{N-1}^{(\text{0-inst.})}} \int_{b\leq\lambda_i\leq a} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} (x-\lambda_i)^2 \Delta_{N-1}(\lambda_1,\cdots,\lambda_{N-1})^2 \times \\
&\quad \times e^{-N\sum_{i=1}^{N-1} V(\lambda_i)}\,, \\
&= \frac{\int_{\text{0-inst.}} d\phi_{N-1}\, \det(x-\phi_{N-1})^2\, e^{-N\,\mathrm{tr}\,V(\phi_{N-1})}}{\int_{\text{0-inst.}} d\phi_{N-1}\, e^{-N\,\mathrm{tr}\,V(\phi_{N-1})}}\,.
\end{aligned} \tag{5.2}$$

Here ϕ_{N-1} is an $(N-1)\times(N-1)$ hermitean matrix.

Keeping (3.6) in mind, we can calculate the factor $Z_{N-1}^{(\text{0-inst.})}/Z^{(\text{0-inst.})}$ in the last line of (5.1) as follows.

$$\frac{Z_{N-1}^{(\text{0-inst.})}}{Z^{(\text{0-inst.})}} = \frac{1}{Nh_{N-1}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) = \frac{r_N}{Nh_N}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \tag{5.3}$$

The $\mathcal{O}(1/N)$ correction comes from the instanton contributions. The calculation of h_N in appendix E of ref. [19] holds for our case. The result is

$$h_N = 2\pi\sqrt{r(1)}\,e^{-NV_{\text{eff}}^{(0)}(a)}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \tag{5.4}$$

We need $\langle \det(x-\phi_{N-1})^2\rangle_{N-1}^{(\text{0-inst.})}$ outside the cut, $x > a$. By setting $x = \lambda_N$ formally, we find

$$\begin{aligned}
\langle \det(x-\phi_{N-1})^2\rangle_{N-1}^{(\text{0-inst.})} &= \frac{1}{(N-1)!h_0h_1\cdots h_{N-2}} \times \\
&\quad \times \int_{b\le\lambda_i\le a} \prod_{i=1}^{N-1} d\lambda_i\, \Delta_N(\lambda_1,\cdots,\lambda_N)^2\, e^{-N\sum_{i=1}^{N-1}V(\lambda_i)} \\
&= P_{N-1}(\lambda_N)^2 + \frac{h_{N-1}}{h_{N-2}}P_{N-2}(\lambda_N)^2 + \cdots + \frac{h_{N-1}}{h_0}P_0(\lambda_N)^2 \\
&= P_{N-1}(x)^2 + r_{N-1}P_{N-2}(x)^2 + \cdots + r_{N-1}r_{N-2}\cdots r_1P_0(x)^2
\end{aligned} \tag{5.5}$$

(5.5) holds exactly if there is no limitation to the 0-instanton sector. Actually, there is a relative $\mathcal{O}(1/N)$ error coming from the instanton contributions in each term in the last line of (5.5). Here we ignored these errors because they turn out to only lead to a relative $\mathcal{O}(1/N)$ correction in the final result. (5.5) is further calculated following ref. [19].

$$\begin{aligned}
\langle \det(x-\phi_{N-1})^2\rangle_{N-1}^{(\text{0-inst.})} &= \left(\frac{k^{(0)}(x,1)}{q(x,1)}\right)^2 \exp\left[2N\int_0^{1-\frac{1}{N}} d\sigma\, \log k^{(0)}(x,\sigma)\right]\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= \left(\frac{k^{(0)}(x,1)}{q(x,1)}\right)^2 \frac{1}{(k^{(0)}(x,1))^2} \exp\left[2N\int_0^{1} d\sigma\, \log k^{(0)}(x,\sigma)\right] \times \\
&\quad \times \left(1+\mathcal{O}\left(\frac{1}{N}\right)\right),
\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
k^{(0)}(x,\sigma) &= \frac{1}{2}\left(x - s(\sigma) + \sqrt{(x-s(\sigma))^2 - 4r(\sigma)}\right), \\
q(x,\sigma) &= \sqrt{(x-s(\sigma))^2 - 4r(\sigma)}\,.
\end{aligned} \tag{5.7}$$

These are $\mathcal{O}(N^0)$ quantities. (5.6) implies that outside the cut

$$V_{\text{eff}}^{(0)}(x) = V(x) - 2\int_0^1 d\sigma\, \log k^{(0)}(x,\sigma)\,. \tag{5.8}$$

This would coincide with (4.7) outside the cut, so that the structure of the cut in (5.8) should agree with that in $W(x)$. This observation leads to a relation

$$(x-s(1))^2 - 4r(1) = (x-a)(x-b)\,, \tag{5.9}$$

from which we obtain

$$x_* = s_c + 2\sqrt{r_c}\,. \tag{5.10}$$

From (5.1), (5.3) and (5.6), we obtain

$$\mathcal{A}^{(n)} = \frac{1}{h_N}\int_{x_n} dx\, \frac{r_N}{(k^{(0)}(x,1))^2}\left(\frac{k^{(0)}(x,1)}{q(x,1)}\right)^2 e^{-NV_{\text{eff}}^{(0)}(x)}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right), \tag{5.11}$$

where h_N and $V_{\text{eff}}^{(0)}(x)$ are given in (5.4) and (5.8), respectively. From this expression, we see that $\log\mathcal{A}^{(n)}$ starts with $\mathcal{O}(N)$ and the saddle point integral in (5.11) actually gives $\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms in $\log\mathcal{A}^{(n)}$.

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First, we assume that $V(x)$ is generic without accidental symmetry. We must treat separately the case in which $V(x)$ is even. Substituting (3.11) and (3.22) into the derivative of (5.8) and using (5.10) leads to

$$V_{\rm eff}^{(0)\prime}(x) = V'(x) - \int_0^{\varepsilon^{-k}\nu^{-1}} d\tau\, \varepsilon^{k-\frac{1}{2}}\nu \left(\frac{1}{\sqrt{r_c \alpha u + x_* \tilde{\zeta}}} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \right). \tag{5.12}$$

As mentioned in section 3, $W(x) = -\frac{1}{2}V_{\rm eff}^{(0)\prime}(x)$ starts with a term proportional to $\varepsilon^{k-\frac{1}{2}}$ in the scaling limit, so that we are allowed to simplify (5.12) as

$$V_{\rm eff}^{(0)\prime}(x) = \nu\alpha^{-\frac{1}{2}} r_c^{-\frac{1}{2}} \varepsilon^{k-\frac{1}{2}} \int^0 d\tau \frac{1}{\sqrt{u(\tau) + \alpha^{-1} r_c^{-\frac{1}{2}} x_* \tilde{\zeta}}} + \mathcal{O}(\epsilon^k), \tag{5.13}$$

where we keep only the contribution from the one end $\tau = 0$ of the integral region. The other terms with integer power in ε lower than $\varepsilon^{k-\frac{1}{2}}$ would be canceled in (5.12). $\partial_{\tilde{\zeta}} V_{\rm eff}^{(0)}(x)$ starts with the $\mathcal{O}\varepsilon^{k+\frac{1}{2}}$ term and $N \sim \varepsilon^{-k-\frac{1}{2}}$, so that $N\partial_{\tilde{\zeta}} V_{\rm eff}^{(0)}(x)$ is finite and we can ignore the $\mathcal{O}(\varepsilon^k)$ correction in (5.13). We also perform the change of the integration variable $u(\tau) = \sqrt{\mu} w$. Then, using (3.12) and (3.14), we obtain

$$\frac{\partial V_{\rm eff}^{(0)}(x)}{\partial \tilde{\zeta}} = \varepsilon^{k+\frac{1}{2}} \nu \alpha^{-\frac{1}{2}} r_c^{-\frac{1}{2}} x_* \sum_{j=1}^{\infty} j t_j \mu^{\frac{j}{2}-\frac{1}{4}} \int^1 dw\, w^{j-1} \frac{1}{\sqrt{w + \alpha^{-1} r_c^{-\frac{1}{2}} x_* \mu^{-\frac{1}{2}} \tilde{\zeta}}}. \tag{5.14}$$

Substituting (3.13) into (5.14) and using the formula (B.1) yields

$$\frac{\partial V_{\rm eff}^{(0)}(x)}{\partial \tilde{\zeta}} = \varepsilon^{k+\frac{1}{2}} \sqrt{\frac{\pi}{2}} \nu \alpha^{\frac{1}{2}} \Omega \mu^{\frac{k}{2}-\frac{1}{4}} \int_{-\frac{\Omega\tilde{\zeta}}{\sqrt{\mu}}}^1 dw \left(w + \frac{\Omega \tilde{\zeta}}{\sqrt{\mu}} \right)^{-\frac{1}{2}} P_{k-1}(w), \tag{5.15}$$

where $\Omega = \alpha^{-1} r_c^{-\frac{1}{2}} x_*$ and P_{k-1} is the Legendre polynomial of degree $k-1$. We have specified consistently the lower end of the integral region in such a way that it does not contribute to the value of the integral. Furthermore, using (B.2), (B.3) and (2.5), we obtain

$$\frac{\partial V_{\rm eff}^{(0)}(x)}{\partial \tilde{\zeta}} = \varepsilon^{k+\frac{1}{2}} \nu \alpha^{\frac{1}{2}} \Omega \frac{(-1)^{k+1}\sqrt{2\pi}}{2k-1} \mu^{\frac{k}{2}-\frac{1}{4}} \sqrt{2} \eta_k(\Omega \mu^{-\frac{1}{2}} \tilde{\zeta}). \tag{5.16}$$

The above calculation that reduces (5.14) to (5.16) is essentially same as that in appendix B in [17]. The lefthand side of (5.16) is proportional to the leading term in the scaling limit of $W(x)$, which is the universal part of the resolvent, so that we obtain

$$\chi_a = \Omega^{-1} = \alpha x_*^{-1} r_c^{\frac{1}{2}}, \qquad \chi_n = \xi_n \chi_a. \tag{5.17}$$

That is, $x = x_n$ corresponds to $\tilde{\zeta} = \Omega^{-1}\mu^{\frac{1}{2}}\xi_n$ and $x = a$ corresponds to $\tilde{\zeta} = -\Omega^{-1}\mu^{\frac{1}{2}}$. By integrating (5.16) over $\tilde{\zeta}$, we finally obtain

$$N V_{\rm eff}^{(0)}(x) = (\kappa^{-1}\nu\alpha^{\frac{1}{2}}) \frac{(-1)^{k+1}\sqrt{2\pi}}{2k-1} g_s^{-1} \mu^{\frac{k}{2}+\frac{1}{4}} v_k(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta}). \tag{5.18}$$

In order to determine the overall factor $\kappa^{-1}\nu\alpha^{\frac{1}{2}}$ in (5.18), we need a physical input. We adopt the sphere amplitude as the physical input. First, we calculate the sphere contribution to the free energy of the matrix model (3.15). By performing the change of the variable $u(\tau) = \sqrt{\mu}w$ and using (3.12), (3.13), (3.14), (B.1) and (B.4), we obtain

$$\begin{aligned} F^{(\text{sphere})} &= \frac{1}{2}(\kappa^{-1}\nu\alpha^{\frac{1}{2}})^2 g_s^{-2}\mu^{k+\frac{1}{2}} \sum_{p=0}^{[\frac{k-1}{2}]}\sum_{q=0}^{[\frac{k}{2}]}(k-2p)C_{k-2p}C_{k-2q}\int^1 dw\, w^{4k-2p-2q} \\ &= \frac{1}{2}(\kappa^{-1}\nu\alpha^{\frac{1}{2}})^2 \Xi_k g_s^{-2}\mu^{k+\frac{1}{2}}\,, \end{aligned} \tag{5.19}$$

where

$$\Xi_k = -\frac{\pi}{2}\frac{1}{(2k+1)(2k-1)(2k-3)}\,. \tag{5.20}$$

If the sphere amplitude is given by

$$F^{(\text{sphere})} = d_k g_s^{-2}\mu^{k+\frac{1}{2}}\,, \tag{5.21}$$

where d_k is a certain universal constant, $\kappa^{-1}\nu\alpha^{\frac{1}{2}}$ is fixed as

$$\kappa^{-1}\nu\alpha^{\frac{1}{2}} = \sqrt{\frac{2d_k}{\Xi_k}}\,. \tag{5.22}$$

Then, $NV_{\text{eff}}^{(0)}(x)$ is determined as

$$NV_{\text{eff}}^{(0)}(x) = D_k g_s^{-1}\mu^{\frac{k}{2}+\frac{1}{4}}v_k(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta})\,, \tag{5.23}$$

where

$$D_k = \frac{2\sqrt{\pi}(-1)^{k+1}}{2k-1}\sqrt{\frac{d_k}{\Xi_k}} = (-1)^{k+1}2\sqrt{2}\sqrt{\frac{(2k+1)(2k-3)}{2k-1}}\sqrt{-d_k}\,. \tag{5.24}$$

We are ready to calculate $A^{(n)}$ given in (5.11).

$$\begin{aligned} \mathcal{A}^{(n)} &= \frac{1}{h_N}\frac{r_N}{(k^{(0)}(x_n,1))^2}\left(\frac{k^{(0)}(x_n,1)}{q(x_n,1)}\right)^2 x_*\varepsilon e^{-D_k g_s^{-1}\mu^{\frac{k}{2}+\frac{1}{4}}v_k(\xi_n)}\times \\ &\quad\times\int d\tilde{\zeta}\, e^{-\frac{D_k g_s^{-1}}{2}\mu^{\frac{k}{2}-\frac{3}{4}}\Omega^2 v_k''(\xi_n)(\tilde{\zeta}-\Omega^{-1}\mu^{\frac{1}{2}}\xi_n)^2} \\ &= \frac{1}{h_N}\frac{r_N}{(k^{(0)}(x_n,1))^2}\left(\frac{k^{(0)}(x_n,1)}{q(x_n,1)}\right)^2 x_*\Omega^{-1}\varepsilon\sqrt{\frac{2\pi}{D_k v_k''(\xi_n)}}\mu^{-\frac{k}{4}+\frac{3}{8}}e^{-D_k g_s^{-1}\mu^{\frac{k}{2}+\frac{1}{4}}v_k(\xi_n)}\,, \end{aligned} \tag{5.25}$$

where we have used (3.22). We see from (5.4), (5.10) and (5.17) that in the scaling limit

$$\begin{aligned} h_N &= 2\pi\sqrt{r_c}e^{-D_k g_s^{-1}\mu^{\frac{k}{2}+\frac{1}{4}}v_k(-1)} = 2\pi\sqrt{r_c} \\ \frac{r_N}{(k^{(0)}(x_n,1))^2} &= 1\,, \\ \left(\frac{k^{(0)}(x_n,1)}{q(x_n,1)}\right)^2 &= \frac{1}{4\varepsilon\alpha\sqrt{\mu}}\frac{1}{\xi_n+1}\,. \end{aligned} \tag{5.26}$$

Thus, noting $\Omega = \alpha^{-1} r_c^{-\frac{1}{2}} x_*$, we finally obtain

$$\mathcal{A}^{(n)} = \frac{1}{8}\sqrt{\frac{2}{\pi D_k v_k''(\xi_n)}}\frac{1}{\xi_n + 1} g_s^{-\frac{1}{2}} \mu^{-\frac{k}{4}-\frac{1}{8}} e^{-D_k g_s^{-1} \mu^{\frac{k}{2}+\frac{1}{4}} v_k(\xi_n)}\,, \tag{5.27}$$

where ξ_n, $v_k(\xi)$, $v_k(\xi_n)$ and D_k are given in (2.7), (2.10), (2.11) and (5.24), respectively. It follows from this expression that $\mathcal{A}^{(n)}$ is indeed universal i.e. independent of the detailed structure in the potential of the matrix model.

Next, let us consider the case in which $V(x)$ is even. s_n vanishes identically in this case. We can see that all the equations (5.12)$\sim$(5.18) also hold for this case if we set $x_* = 2\sqrt{r_c}$ and replace α with $\alpha/2$. Namely,

$$NV_{\text{eff}}^{(0)}(x) = (\kappa^{-1}\nu\alpha^{\frac{1}{2}})\frac{(-1)^{k+1}\sqrt{\pi}}{2k-1} g_s^{-1}\mu^{\frac{k}{2}+\frac{1}{4}} v_k(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta}) \tag{5.28}$$

with $\Omega = 4\alpha^{-1}$ and $\chi_a = \frac{1}{4}\alpha$. $F^{(\text{sphere})}$ given in (3.15) does not include s_n so that (5.19) is invariant. However, instead of (5.21) we must fix $F^{(\text{sphere})}$ as

$$F^{(\text{sphere})} = 2d_k g_s^{-2}\mu^{k+\frac{1}{2}}\,, \tag{5.29}$$

since due to the Z_2 symmetry there are two critical points each of which contributes equally to the free energy. Hence, $\kappa^{-1}\nu\alpha^{\frac{1}{2}}$ in (5.28) is determined as

$$\kappa^{-1}\nu\alpha^{\frac{1}{2}} = 2\sqrt{\frac{d_k}{\Xi_k}}\,. \tag{5.30}$$

This indeed reduces (5.28) to (5.23). (5.25) and (5.26) also hold for this case if we set $x_* = 2\sqrt{r_c}$ and replace α with $\frac{\alpha}{2}$. Noting $\Omega = 4\alpha^{-1}$, we can easily see that $A^{(n)}$ is indeed given by (5.27).

Thus, the proof of the universality of $A^{(n)}$ is completed. Note that $A^{(n)}$ is pure imaginary for odd n and real for even n. This reflects the fact that $\xi = \xi_n$ with n odd corresponds to a local maximum and $\xi = \xi_n$ with n even corresponds to a local minimum.

As a check on our calculation, let us calculate $A^{(1)}$ in the $k = 2$ case, which would coincide with μ in [19]. The normalization of $F^{(\text{sphere})}$ in [19] corresponds to $d_2 = -\frac{4}{15}$. Using (2.7), (2.10), (2.11) and (5.24), we can calculate the quantities that appears in (5.27) with $k = 2$ and $n = 1$ as

$$D_2 = -\frac{4\sqrt{2}}{3}\,, \qquad \xi_1 = \frac{1}{2}\,, \qquad v_2(\xi_1) = -\frac{3\sqrt{6}}{5}\,, \qquad v_2''(\xi_1) = \sqrt{6}\,. \tag{5.31}$$

By substituting these quantities into (5.27), we obtain

$$\mathcal{A}^{(1)} = \frac{i}{8\cdot 3^{\frac{3}{4}}\sqrt{\pi}} g_s^{-\frac{1}{2}}\mu^{-\frac{5}{8}} e^{-\frac{8\sqrt{3}}{5} g_s^{-1}\mu^{\frac{5}{4}}}\,. \tag{5.32}$$

This indeed coincides with μ in [19].

Finally, let us see that the leading order of $\log \mathcal{A}^{(n)}$ indeed agrees with the $(1,n)$ ZZ brane disk amplitude, which we denote by Z_n. The leading order of $\log \mathcal{A}^{(n)}$ is given by

$$-D_k g_s^{-1} \mu^{\frac{k}{2}+\frac{1}{4}} v_k(\xi_n)\,. \tag{5.33}$$

We can evaluate Z_n in Liouville theory by using eqs.(B.4) and (B.6) in [20] as

$$Z_n = (-1)^n \frac{4 \cdot 2^{\frac{5}{4}}}{\pi^{\frac{3}{2}}} \frac{(2k-1)^{\frac{1}{4}}}{2k+1} \frac{\Gamma(\frac{2k-3}{2k-1})}{\Gamma(\frac{2k-1}{2})} \left(\sin \frac{2\pi}{2k-1}\right)^{\frac{1}{2}} \sin \frac{2\pi n}{2k-1} g_s^{-1} \mu^{\frac{k}{2}+\frac{1}{4}}\,. \tag{5.34}$$

We can also calculate the sphere amplitude in Liouville theory by integrating twice the two-point function of the cosmological constant operators which is given in (2.26) in [16]. The result corresponds to[2]

$$d_k = -\frac{1}{\pi^3} \sqrt{\frac{2}{2k-1}} \frac{2k-3}{2k+1} \sin \frac{2\pi}{2k-1} \left(\frac{\Gamma(\frac{2k-3}{2k-1})}{\Gamma(\frac{2k-1}{2})} \right)^2 . \tag{5.35}$$

It is easy to verify that plugging (5.35) into (5.33) actually yields (5.34).

The fact that the leading order of $\log \mathcal{A}^{(n)}$ is the $(1,n)$ ZZ brane disk amplitude is already pointed out in [17]. What is new in this section is that we showed that both the leading order term ($\mathcal{O}(1/g_s)$) and, in particular, the next to leading order terms ($\mathcal{O}(\log g_s)$ and $\mathcal{O}(g_s^0)$) in $\log \mathcal{A}^{(n)}$ are universal. We also showed that the normalization of the ZZ brane disk amplitude is also reproduced precisely by matching the sphere amplitude in the matrix model with that in Liouville theory.

6. 2-instanton sectors and the annulus amplitudes between the ZZ branes

In this section, we calculate $\mathcal{A}^{(n,n')}$. The estimation of the order in the $1/N$ expansion proceeds in the same way as that in the calculation of $\mathcal{A}^{(n)}$. We will not dwell on it in this section. $\log \mathcal{A}^{(n,n')}$ also starts with $\mathcal{O}(N)$. We will evaluate $\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms in $\mathcal{A}^{(n,n')}$. We will see that these terms are universal. We will also show that in the $n \neq n'$ case the leading order term in $\log \mathcal{A}_c^{(n,n')}$, which is $\mathcal{O}(N^0)$, reproduces the annulus amplitude between the $(1,n)$ and $(1,n')$ ZZ branes, where $\mathcal{A}_c^{(n,n')} = \mathcal{A}^{(n,n')}/\mathcal{A}^{(n)}\mathcal{A}^{(n')}$. First, we consider the case in which $n \neq n'$ case. In this case $\mathcal{A}^{(n,n')}$ is given by

$$\begin{aligned}
\mathcal{A}^{(n,n')} &= \frac{Z^{\{0,\cdots,0,q_n=1,0,\cdots,0,q_{n'}=1,0,\cdots,0\}}}{Z^{(\text{0-inst.})}} \\
&= \frac{1}{Z^{(\text{0-inst.})}} N(N-1) \int_{x_n} dx \int_{x_{n'}} dy \int_{b \leq \lambda_i \leq a} \prod_{i=1}^{N-2} d\lambda_i \times
\end{aligned}$$

[2] The relation between our cosmological constant μ and the cosmological constant μ_L in [16] is $\mu = \mu_L \pi \frac{\Gamma(\frac{2k-3}{2k-1})}{\Gamma(\frac{2}{2k-1})}$. Note also that for odd k the overall sign of d_k obtained from (2.26) in [16] is different from that in (5.35). It seems possible to attribute this difference to the ambiguity of the sign of the norm in nonunitary models. We adjust the overall sign of d_k in (5.35) to minus in such a way that it is consistent with the result in the matrix model.

$$
\begin{aligned}
&\times(x-y)^2\left(\prod_{i=1}^{N-2}(x-\lambda_i)^2(y-\lambda_i)^2\right)\Delta_{N-2}(\lambda_1,\cdots,\lambda_{N-2})^2\times\\
&\times e^{-N\sum_{i=1}^{N-2}V(\lambda_i)}\,e^{-NV(x)-NV(y)}\\
=\ &N(N-1)\frac{Z_{N-2}^{(\text{0-inst.})}}{Z^{(\text{0-inst.})}}\int_{x_n}dx\int_{x_{n'}}dy\,\langle\det(x-\phi_{N-2})^2\det(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}\times\\
&\times e^{-NV(x)-NV(y)+\log(x-y)^2}\,,
\end{aligned}\tag{6.1}
$$

where

$$
\begin{aligned}
Z_{N-2}^{(\text{0-inst.})}&=\int_{b\le\lambda_i\le a}\prod_{i=1}^{N-2}d\lambda_i\,\Delta_{N-2}(\lambda_1,\cdots,\lambda_{N-2})^2\,e^{-N\sum_{i=1}^{N-2}V(\lambda_i)}\,,\\
\langle\det(x-\phi_{N-2})^2&\det(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}=\\
&=\frac{1}{Z_{N-2}^{(\text{0-inst.})}}\int_{b\le\lambda_i\le a}\prod_{i=1}^{N-2}d\lambda_i\,\left(\prod_{i=1}^{N-2}(x-\lambda_i)^2(y-\lambda_i)^2\right)\Delta_{N-2}(\lambda_1,\cdots,\lambda_{N-2})^2\times\\
&\quad\times e^{-N\sum_{i=1}^{N-2}V(\lambda_i)}\,,\\
&=\frac{\int_{\text{0-inst.}}d\phi_{N-2}\,\det(x-\phi_{N-2})^2\det(y-\phi_{N-2})^2\,e^{-N\,\mathrm{tr}\,V(\phi_{N-2})}}{\int_{\text{0-inst.}}d\phi_{N-2}\,e^{-N\,\mathrm{tr}\,V(\phi_{N-2})}}\,.
\end{aligned}\tag{6.2}
$$

Here ϕ_{N-2} is an $(N-2)\times(N-2)$ hermitean matrix. The factor $Z_{N-2}^{(\text{0-inst.})}/Z^{(\text{0-inst.})}$ in the last line of (6.1) is calculated as

$$
\frac{Z_{N-2}^{(\text{0-inst.})}}{Z^{(\text{0-inst.})}}=\frac{1}{N(N-1)h_{N-1}h_{N-2}}=\frac{r_N^2r_{N-1}}{N(N-1)h_N^2}\,.\tag{6.3}
$$

$\langle\det(x-\phi_{N-2})^2\det(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}$ is evaluated as follows.

$$
\begin{aligned}
\langle\det(x-\phi_{N-2})^2&\det(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}=\\
&=\langle e^{\mathrm{tr}\log(x-\phi_{N-2})^2+\mathrm{tr}\log(y-\phi_{N-2})^2}\rangle_{N-2}^{(\text{0-inst.})}\\
&=\exp\Big[\langle\mathrm{tr}\log(x-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}+\langle\mathrm{tr}\log(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}+\\
&\qquad+\frac{1}{2}\langle(\mathrm{tr}\log(x-\phi_{N-2})^2)^2\rangle_{N-2,c}^{(\text{0-inst.})}+\frac{1}{2}\langle(\mathrm{tr}\log(y-\phi_{N-2})^2)^2\rangle_{N-2,c}^{(\text{0-inst.})}+\\
&\qquad+\langle\mathrm{tr}\log(x-\phi_{N-2})^2\,\mathrm{tr}\log(y-\phi_{N-2})^2\rangle_{N-2,c}^{(\text{0-inst.})}+\cdots\Big]\\
&=\langle\det(x-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}\langle\det(y-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}\times\\
&\quad\times\exp\Big[\langle\mathrm{tr}\log(x-\phi_{N-2})^2\,\mathrm{tr}\log(y-\phi_{N-2})^2\rangle_{N-2,c}^{(\text{0-inst.})}+\cdots\Big]\,,
\end{aligned}\tag{6.4}
$$

where the subscript 'c' stands for the connected part. We can calculate $\langle\det(x-\phi_{N-2})^2\rangle_{N-2}^{(\text{0-inst.})}$ in a way similar to (5.5) and (5.6) as

$$
\langle\det(x-\phi_{N-1})^2\rangle_{N-2}^{(\text{0-inst.})}=\left(\frac{k^{(0)}(x,1)}{q(x,1)}\right)^2\frac{1}{(k^{(0)}(x,1))^4}\exp\left[2N\int_0^1d\sigma\,\log k^{(0)}(x,\sigma)\right].\tag{6.5}
$$

A similar expression holds for $\langle \det(y-\phi_{N-2})^2\rangle^{(0\text{-inst.})}_{N-2}$. In the leading order of the $1/N$ expansion, we can set

$$\langle \mathrm{tr}\log(x-\phi_{N-2})^2\,\mathrm{tr}\log(y-\phi_{N-2})^2\rangle^{(0\text{-inst.})}_{N-2,c} = 4\langle \mathrm{tr}\log(x-\phi)\,\mathrm{tr}\log(y-\phi)\rangle_c\,. \tag{6.6}$$

The righthand side is calculated in appendix B. From (6.1), (6.3), (6.4), (6.5) and (6.6), we obtain

$$\begin{aligned}\mathcal{A}^{(n,n')} = \frac{r_N^2 r_{N-1}}{h_N^2}\int_{x_n} dx\int_{x_{n'}} dy\left(\frac{k^{(0)}(x,1)}{q(x,1)}\right)^2 \frac{1}{(k^{(0)}(x,1))^4}\left(\frac{k^{(0)}(y,1)}{q(y,1)}\right)^2\frac{1}{(k^{(0)}(y,1))^4}\times\\ \times\exp\left[-NV^{(0)}_{\text{eff}}(x)-NV^{(0)}_{\text{eff}}(y)+\log(x-y)^2+4\langle \mathrm{tr}\log(x-\phi)\,\mathrm{tr}\log(y-\phi)\rangle_c\right]\end{aligned} \tag{6.7}$$

By noting that in the scaling limit

$$r_N^2 r_{N-1}\left(\frac{1}{k^{(0)}(x_n,1)}\right)^4\left(\frac{1}{k^{(0)}(x_{n'},1)}\right)^4 = \frac{1}{r_c} \tag{6.8}$$

and recalling the calculation of $\mathcal{A}^{(n)}$, we find

$$\begin{aligned}\mathcal{A}^{(n,n')} &= \mathcal{A}^{(n)}\mathcal{A}^{(n')}\mathcal{A}_c^{(n,n')}\,,\\ \mathcal{A}_c^{(n,n')} &= \exp\left[4\langle \mathrm{tr}\log(x_n-\phi)\,\mathrm{tr}\log(x_{n'}-\phi)\rangle_c+\log(x_n-x_{n'})^2-\log r_c\right].\end{aligned} \tag{6.9}$$

Here $\mathcal{A}_c^{(n,n')}$ is interpreted as the 'connected part' of $\mathcal{A}^{(n,n')}$ since $\mathcal{A}^{(n)}\mathcal{A}^{(n')}$ is the product of the 1-instanton contributions. As seen in the previous section, the leading order of $\log\mathcal{A}^{(n)}$ is the $(1,n)$ ZZ brane disk amplitude, so that the leading order of $\log\mathcal{A}_c^{(n,n')}$ is expected to be the annulus amplitude between the $(1,n)$ and $(1,n')$ ZZ branes. In the following, we will show that this is indeed the case. Using (B.6) leads to

$$\begin{aligned}\log\mathcal{A}_c^{(n,n')} &= 4\langle \mathrm{tr}\log(x_n-\phi)\,\mathrm{tr}\log(x_{n'}-\phi)\rangle_c+\log(x_n-x_{n'})^2-\log r_c\\ &= \log\Big(2x_nx_{n'}-(a+b)(x_n+x_{n'})+a^2+b^2-\\ &\qquad -2\sqrt{(x_n-a)(x_n-b)(x_{n'}-a)(x_{n'}-b)}\Big)^2-\\ &\quad -\log\left(\sqrt{(x_n-a)(x_n-b)}+\sqrt{(x_{n'}-a)(x_{n'}-b)}\right)^4+\\ &\quad +\log\left(\sqrt{x_n-a}+\sqrt{x_n-b}\right)^4+\log\left(\sqrt{x_{n'}-a}+\sqrt{x_{n'}-b}\right)^4-\\ &\quad -\log(a-b)^4-\log 16+\log(x_n-x_{n'})^2-\log r_c\,.\end{aligned} \tag{6.10}$$

Recalling that in the scaling limit,

$$\begin{aligned}a &= x_*(1-\chi_a\sqrt{\mu}\varepsilon)\,,\\ x_n &= x_*(1+\chi_a\xi_n\sqrt{\mu}\varepsilon)\,,\end{aligned} \tag{6.11}$$

we calculate the quantities that appears in (6.10).

$$\begin{aligned}
2x_n x_{n'} - (a+b)(x_n + x_{n'}) + a^2 + b^2 - \\
-2\sqrt{(x_n-a)(x_n-b)(x_{n'}-a)(x_{n'}-b)} &= (x_*-b)^2(1+\mathcal{O}(\varepsilon))\,, \\
\sqrt{(x_n-a)(x_n-b)} + \sqrt{(x_{n'}-a)(x_{n'}-b)} &= \sqrt{x_*-b}\left(\sqrt{x_n-a}+\sqrt{x_{n'}-a}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{2}}))\,, \\
\sqrt{x_n-a}+\sqrt{x_n-b} &= \sqrt{x_*-b}\,(1+\mathcal{O}(\varepsilon^{\frac{1}{2}}))\,, \\
\sqrt{x_{n'}-a}+\sqrt{x_{n'}-b} &= \sqrt{x_*-b}(1+\mathcal{O}\,(\varepsilon^{\frac{1}{2}}))\,, \\
a-b &= (x_*-b)(1+\mathcal{O}(\varepsilon))\,. \qquad (6.12)
\end{aligned}$$

Substituting these into (6.15) yields

$$\log \mathcal{A}_c^{(n,n')} = \log\left[\frac{(x_n - x_{n'})^2}{(\sqrt{x_n-a}+\sqrt{x_{n'}-a})^4}\frac{(x_*-b)^2}{16r_c}\right] + \mathcal{O}(\varepsilon^{\frac{1}{2}})\,. \qquad (6.13)$$

Furthermore, by using (6.11) and a relation

$$r_c = \frac{(x_*-b)^2}{16}(1+\mathcal{O}(\varepsilon)) \qquad (6.14)$$

which follows from (5.9), we finally obtain

$$\log \mathcal{A}_c^{(n,n')} = \log\frac{(\xi_n - \xi_{n'})^2}{(\sqrt{\xi_n+1}+\sqrt{\xi_{n'}+1})^4}\,. \qquad (6.15)$$

This is universal. The model-dependent quantities such as r_c are indeed canceled. It follows that $\mathcal{A}^{(n,n')}$ is also universal because $\mathcal{A}^{(n)}$ and $\mathcal{A}^{(n')}$ are universal.

Let us see that $\mathcal{A}^{(n,n')}$ with $n \neq n'$ is the annulus amplitude between the ZZ branes in Liouville theory. First, we rewrite $z_n^{\pm}$ by ξ_n.

$$z_n^{\pm} = -\sin\frac{\pm\pi n}{2k-1} = \mp\sqrt{\frac{\xi_n+1}{2}}\,. \qquad (6.16)$$

Using this, we express the annulus amplitude between the $(1,n)$ and $(1,n')$ ZZ branes (2.12) in terms of ξ_n:

$$Z_{n,n'} = \log\frac{(\xi_n - \xi_{n'})^2}{(\sqrt{\xi_n+1}+\sqrt{\xi_{n'}+1})^4}\,. \qquad (6.17)$$

This indeed agrees with $\log \mathcal{A}_c^{(n,n')}$.

Next, we consider the case in which $n = n'$. The same calculation as the $n \neq n'$ case leads to

$$\begin{aligned}
\mathcal{A}^{(n,n)} &= \frac{1}{2}(\mathcal{A}^{(n)})^2 \exp\left[4\langle \operatorname{tr}\log(x_n-\phi)\operatorname{tr}\log(x_n-\phi)\rangle_c + \log(x_n-x_n)^2 - \log r_c\right] \\
&= \frac{1}{2}(\mathcal{A}^{(n)})^2\frac{(\xi_n-\xi_n)^2}{(\sqrt{\xi_n+1}+\sqrt{\xi_n+1})^4}\,. \qquad (6.18)
\end{aligned}$$

That is, $\mathcal{A}^{(n,n)}$ vanishes and $\log\mathcal{A}^{(n,n)}$ diverges. This is consistent with the result in Liouville theory where $Z_{n,n}$ also diverges.[3]

[3]It is already pointed out in [20] that the double zero of $e^{Z_{nn}}$ stems from the Vandermonde determinant of the matrix model.

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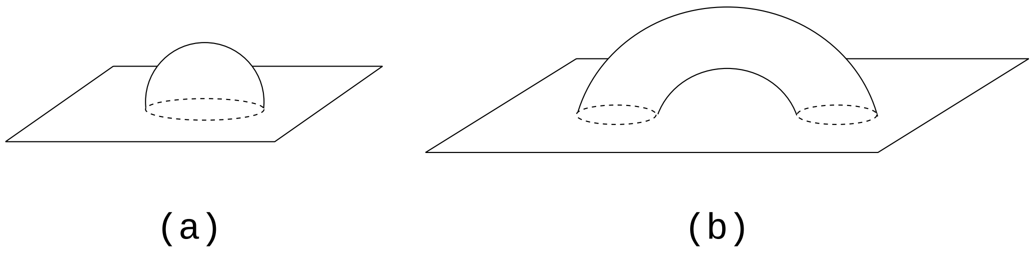

Figure 2: $\log \mathcal{A}^{(n)}$: (a) the leading order (b) the next to leading orders.

7. Summary and discussion

In this paper, we analyzed the k-th multicritical region of the one matrix model that corresponds to the $(2, 2k-1)$ minimal string theory. We divided the partition function of the matrix model in terms of the instanton numbers. We evaluated the ratio of the 1-instanton sector to the 0-instanton sector, $\mathcal{A}^{(n)}$, and the ratio of the 2-instanton sector to the 0-instanton sector, $\mathcal{A}^{(n,n')}$. We found that $\log \mathcal{A}^{(n)}$ and $\log \mathcal{A}^{(n,n')}$ start with $\mathcal{O}(N)$ terms. We calculated the $\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms in $\log \mathcal{A}^{(n)}$ and $\log \mathcal{A}^{(n,n')}$, which correspond to $\mathcal{O}(1/g_s)$, $\mathcal{O}(\log g_s)$ and $\mathcal{O}(g_s^0)$ terms in the continuum limit, respectively. Our results are as follows. (i)$\mathcal{O}(N)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$ terms in $\log \mathcal{A}^{(n)}$ and $\log \mathcal{A}^{(n,n')}$ are universal i.e. independent of the detailed structure in the potential of the matrix model. (ii)The $\mathcal{O}(N)$ term in $\log \mathcal{A}^{(n)}$ is equal to the $(1,n)$ ZZ brane disk amplitude, which is proportional to $1/g_s$. (iii)When $\mathcal{A}^{(n,n')}$ with $n \neq n'$ is expressed as $\mathcal{A}^{(n,n')} = \mathcal{A}^{(n)}\mathcal{A}^{(n')}\mathcal{A}_c^{(n,n')}$, $\log \mathcal{A}_c^{(n,n')}$ starts with $\mathcal{O}(N^0)$ term. This term reproduces the annulus amplitude between the $(1,n)$ and $(1,n')$ ZZ branes in Liouville theory. (iv)The $\mathcal{O}(N)$ term in $\log \mathcal{A}^{(n,n)}$ are given by that in $\log \mathcal{A}^{(n)}$ while one of the $\mathcal{O}(N^0)$ terms in $\log \mathcal{A}^{(n,n)}$ diverges. This makes $\log \mathcal{A}^{(n,n)}$ vanish and is consistent with the result in Liouville theory.

The above results allow us to assign figures 1a and 3 to the leading orders of $\log \mathcal{A}^{(n)}$ and $\log \mathcal{A}_c^{(n,n')}$, respectively. We express $\mathcal{A}^{(n,n)}$ as $\mathcal{A}^{(n,n)} = \frac{1}{2}(\mathcal{A}^{(n)})^2\mathcal{A}_c^{(n,n)}$. Then, it is natural to assign figure 1b to the next to leading orders ($\mathcal{O}(\log N)$ and $\mathcal{O}(N^0)$) of $\log \mathcal{A}^{(n)}$ and figure 2 to the leading order of $\mathcal{A}_c^{(n,n)}$, which is divergent. Namely, the next to leading orders of $\log \mathcal{A}^{(n)}$ can be interpreted as the annulus stretched within a single $(1,n)$ ZZ brane while the leading order of $\log \mathcal{A}^{(n,n)}$ can be interpreted as the annulus stretched from one $(1,n)$ ZZ brane to the other $(1,n)$ ZZ brane. In Liouville theory, these two diagrams are not distinguished because the ZZ branes have no intrinsic parameter such as the position except the label n, so that the annulus amplitude between the two identical $(1,n)$ ZZ branes in Liouville theory is consistently divergent. As is stressed in [19] in the $k = 2$ case, a nontrivial thing is that the next to leading order of $\log \mathcal{A}^{(n)}$ is a finite and universal quantity, which cannot be evaluated at least so far in Liouville theory. $\mathcal{A}^{(n)}$ is interpreted in the dilute gas approximation as the chemical potential of the n-th instanton as we will see, so that it is suggested that the matrix model possesses the information on the nonperturbative effect that Liouville theory cannot predict. Note that as shown in [19] this quantity cannot be calculated through the loop equation (the string field theory), either.

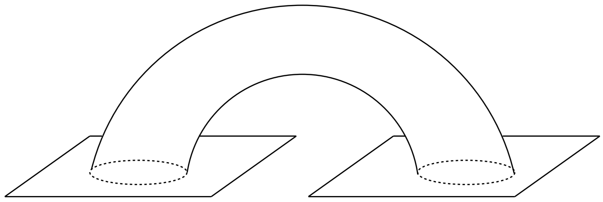

Figure 3: The leading order of $\log \mathcal{A}_c^{(n,n)}$.

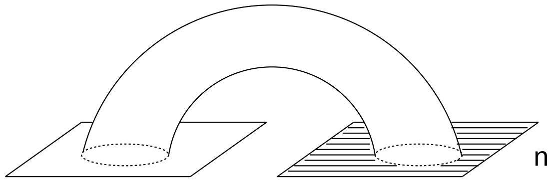

Figure 4: The leading order of $\log \mathcal{A}_c^{(n,n')}$.

Our results imply that $\mathcal{A}^{\{q_1,\cdots,q_{k-1}\}}$ vanishes if $q_n \geq 2$ at least for a certain n so that the expansion of the partition function is terminated with 2^{k-1} terms as

$$Z = Z^{(0\text{-inst.})} \sum_{q_1,\ldots,q_{k-1}=0,1} \mathcal{A}^{\{q_1,\ldots,q_{k-1}\}} . \tag{7.1}$$

The maximum of the total instanton number is $q = k - 1$. Actually, $\mathcal{A}^{\{1,1,\ldots,1\}}$ is nonvanishing. Our results suggest that each $\mathcal{A}^{\{q_1,\cdots,q_{k-1}\}}$ in (7.1) is universal and its 'connected part' systematically reproduces the amplitudes among the ZZ branes in Liouville theory. It is interesting to calculate multi-point amplitudes among the ZZ branes in Liouville theory if possible and compare them with $\mathcal{A}^{\{q_1,\cdots,q_{k-1}\}}$ in (7.1). Note that (7.1) does not contradict the prediction of the string equation in the $k = 2$ case, where the deviation from the perturbative solution for the free energy $F = \log Z$ behaves as $\sim e^{-c/g_s}$ at leading order of e^{-c/g_s}, because at this order the logarithm of Z in (7.1) takes the form of (7.3) below. The reason why $\mathcal{A}^{\{q_1,\cdots,q_{k-1}\}}$ with $q_n \geq 2$ at least for a certain n vanishes is that one cannot place more than two eigenvalues at the same point because of the repulsive force coming from the Vandermonde determinant. However, taking into account the factor $\ln(x_n - x_{n'})^2$ stemming from the Vandermonde determinant from the beginning, one can obtain a 'classical solution' that realizes the situation in which those eigenvalues are separated a little from each other around a local extreme of the potential. Perhaps one can construct a systematic expansion of the partition function based on this classical solution, which differs from the expansion in this paper. This expansion would lead to the singularity destroying deformation argued in [20], which changes the singularity to a cut.

On the other hand, in the dilute gas approximation the interaction between the instantons is turned off. In other words, one ignores $\ln(x_n - x_{n'})^2$ and $\langle \mathrm{tr} \ln(x_n - \phi)\, \mathrm{tr} \ln(x_{n'} - \phi)\rangle_c$ and so on. Hence, in this approximation, the expansion of the partition function obviously reduces to

$$Z = Z^{(0\text{-inst.})} \sum_{q=0}^{\infty} \sum_{q_1+\cdots+q_{k-1}=q} \frac{1}{q_1! \cdots q_{k-1}!} (\mathcal{A}^{(1)})^{q_1} \cdots (\mathcal{A}^{(k-1)})^{q_{k-1}}$$

$$= Z^{(\text{0-inst.})} e^{\mathcal{A}^{(1)}+\cdots+\mathcal{A}^{(k-1)}} . \tag{7.2}$$

Or equivalently,

$$F = F^{(\text{0-inst.})} + \mathcal{A}^{(1)} + \cdots + \mathcal{A}^{(k-1)} . \tag{7.3}$$

This means that the chemical potential of the n-th instanton is $\mathcal{A}^{(n)}$.

A comment is in order. As is pointed in [17, 20], the sign of $NV_{\text{eff}}^{(0)}(\xi_n)$ is $(-1)^{n+1}$ so that $\mathcal{A}^{(n)}$ with n even behaves as $\sim e^{+\frac{1}{g_s}}$, which is catastrophic. The the energy of n-th instanton with n even which corresponds to the local minimum is below the Fermi level. Therefore, the perturbative vacuum is unstable due to the eigenvalues tunneling to these local minima. This is due to the nonunitary nature of the model. Note that the fact that in the $(2,3)$ case n takes only 1 is consistent with the unitarity of the $(2,3)$ model. Thus, the expansion of the partition function of the one matrix model in the instanton numbers should be understood as a formal one in this sense.

It is important to generalize our analysis to the two matrix model, which can represent the unitary noncritical string theories.

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A. The $(2,5)$ model

In this appendix, as an example, we analyze in detail the one matrix model in the 3rd critical region ($k = 3$), which corresponds to the $(2,5)$ minimal string theory. We adopt an even potential of the one matrix model,

$$V(x) = \frac{1}{2}x^2 - \frac{1}{4}g_4 x^4 - \frac{1}{6}g_6 x^6 . \tag{A.1}$$

The first equation in (3.8) gives in the large-N limit

$$\sigma = r(\sigma) - 3g_4 r(\sigma)^2 - 10 g_6 r(\sigma)^3 , \tag{A.2}$$

while the second one is trivial. Correspondingly, $s(\sigma)$ vanishes identically. The scaling limit (3.11) is given in this case by

$$\begin{aligned} g_4 &= \frac{1}{9}(1 - \beta\mu\varepsilon^2) , \\ g_6 &= -\frac{1}{270}(1 - 3\beta\mu\varepsilon^2) , \\ r(\sigma) &= 3\left(1 - \frac{1}{2}\alpha\varepsilon u(\tau)\right) , \\ \sigma &= 1 - \varepsilon^3 \nu\tau , \\ \frac{1}{N} &= \varepsilon^{\frac{7}{2}} \kappa g_s . \end{aligned} \tag{A.3}$$

Plugging (A.3) into (A.2) gives

$$\tau = -\frac{3\beta\alpha}{2\nu}\mu u + \frac{\alpha^3}{8\nu}u^3 \,. \tag{A.4}$$

Comparing this equation with the string equation (3.12), we find

$$\frac{\alpha^3}{8\nu} = \sqrt{\frac{\pi}{8}}\,, \qquad \frac{3\beta\alpha}{2\nu} = \sqrt{\frac{\pi}{8}}\,, \tag{A.5}$$

which are equivalent to

$$\alpha = 2\sqrt{3}\beta^{\frac{1}{2}}\,, \qquad \nu = \sqrt{\frac{8}{\pi}}3\sqrt{3}\beta^{\frac{3}{2}}\,. \tag{A.6}$$

$W(x)$ takes the form

$$W(x) = \frac{1}{2}g_6(x^2 - x_1^2)(x^2 - x_2^2)\sqrt{x^2 - a^2}\,. \tag{A.7}$$

The condition that $R(x) \sim \frac{1}{x}$ when $x \to \infty$ is equivalent to

$$\begin{aligned} g_4 + g_6\left(\frac{1}{2}a^2 + x_1^2 + x_2^2\right) &= 0\,, \\ 1 - g_6\left(\frac{1}{8}a^4 - \frac{1}{2}a^2(x_1^2 + x_2^2) - x_1^2x_2^2\right) &= 0\,, \\ -g_6\left(\frac{1}{32}a^6 - \frac{1}{16}a^4(x_1^2 + x_2^2) + \frac{1}{4}a^2x_1^2x_2^2\right) &= 1\,. \end{aligned} \tag{A.8}$$

In the scaling limit (A.3), a, x_1 and x_2 are determined by these equations as

$$\begin{aligned} a &= x_*\left(1 - \frac{\sqrt{3}}{2}\sqrt{\beta\mu\varepsilon}\right), \\ x_1 &= x_*\left(1 + \frac{\sqrt{3}}{2}\frac{1-\sqrt{5}}{4}\sqrt{\beta\mu\varepsilon}\right), \\ x_2 &= x_*\left(1 + \frac{\sqrt{3}}{2}\frac{1+\sqrt{5}}{4}\sqrt{\beta\mu\varepsilon}\right), \end{aligned} \tag{A.9}$$

where $x_* = 2\sqrt{3}$. If we put $x = x_*(1 + \varepsilon\tilde{\zeta})$, we obtain

$$\begin{aligned} W(x) &= -\frac{\sqrt{2}}{135}\varepsilon^{\frac{5}{2}}x_*^5\chi_a^{\frac{5}{2}}\mu^{\frac{5}{4}}(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta} - \xi_1)(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta} - \xi_2)\sqrt{\Omega\mu^{-\frac{1}{2}}\tilde{\zeta} + 1} \\ &= -\frac{\sqrt{2}}{540}\varepsilon^{\frac{5}{2}}x_*^5\chi_a^{\frac{5}{2}}\mu^{\frac{5}{4}}\sqrt{2}\eta_3(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta}) \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} \chi_a &= \frac{\sqrt{3}}{2}\sqrt{\beta} = \frac{1}{4}\alpha\,, \\ \Omega &= \frac{1}{\chi_a}\,, \\ \xi_1 &= \frac{1-\sqrt{5}}{4} = -\cos\frac{2\pi}{5}\,, \\ \xi_2 &= \frac{1+\sqrt{5}}{4} = -\cos\frac{4\pi}{5}\,. \end{aligned} \tag{A.11}$$

By integrating $W(x)$ over x, we obtain

$$\begin{aligned} NV^{(0)}_{\text{eff}}(x) = -2N\int^x dx'\, W(x') &= \frac{\sqrt{2}}{270} g_s^{-1}\kappa^{-1}x_*^6\chi_a^{\frac{5}{2}}\Omega^{-1}\mu^{\frac{7}{4}}v_3(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta}) \\ &= \frac{4\cdot 3^{\frac{7}{4}}}{5} g_s^{-1}\kappa^{-1}\beta^{\frac{7}{4}}\mu^{\frac{7}{4}}v_3(\Omega\mu^{-\frac{1}{2}}\tilde{\zeta})\,. \end{aligned} \tag{A.12}$$

It is easy to verify that the last line in (A.12) can also be obtained by plugging $k = 3$ and (A.6) into (5.28).

B. Useful formulae

In this appendix, we gather some formulae, which we use in sections 5 and 6.

The following formulae for the Legendre polynomials are used in section 5.

$$P_n(x) = \frac{1}{2^n}\sum_{p=0}^{[\frac{n}{2}]}\frac{(-1)^p(2n-2p)!}{(n-2p)!p!(n-p)!}x^{n-2p}\,. \tag{B.1}$$

$$P_n(-x) = (-1)^n P_n(x)\,. \tag{B.2}$$

$$\int_{-1}^{x} dt\,(x-t)^{-\frac{1}{2}}P_n(t) = \frac{1}{n+\frac{1}{2}}\frac{T_n(x)+T_{n+1}(x)}{\sqrt{x+1}}\,. \tag{B.3}$$

$$\int_0^1 dx\, x^m P_n(x) = \frac{m(m-1)\cdots(m-n+2)}{(m+n+1)(m+n-1)\cdots(m-n+3)}\,. \tag{B.4}$$

The cylinder contribution to the two macroscopic loop correlators in the one matrix model with a general potential is obtained in [22]. The result is

$$\left\langle \operatorname{tr}\left(\frac{1}{x-\phi}\right)\operatorname{tr}\left(\frac{1}{y-\phi}\right)\right\rangle_c = \frac{1}{2(x-y)^2}\left(\frac{xy-\frac{1}{2}(a+b)(x+y)+ab}{\sqrt{(x-a)(x-b)(y-a)(y-b)}}-1\right). \tag{B.5}$$

By integrating this over x and y, we obtain

$$\begin{aligned} &\langle \operatorname{tr}\log(x-\phi)\ \operatorname{tr}\log(y-\phi)\rangle_c = \\ &= \int_x^\infty dx' \int_y^\infty dy' \left\langle \operatorname{tr}\left(\frac{1}{x'-\phi}\right)\operatorname{tr}\left(\frac{1}{y'-\phi}\right)\right\rangle_c \\ &= \frac{1}{2}\log\left|2xy-(a+b)(x+y)+a^2+b^2-2\sqrt{(x-a)(x-b)(y-a)(y-b)}\right| - \\ &\quad -\log\left(\sqrt{(x-a)(x-b)}+\sqrt{(y-a)(y-b)}\right) + \\ &\quad +\log\left(\sqrt{x-a}+\sqrt{x-b}\right)+\log\left(\sqrt{y-a}+\sqrt{y-b}\right)-\log(a-b)-\log 2\,. \end{aligned} \tag{B.6}$$

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